Bell-Like and Peak-Like Loop Solitons in (2+1)-Dimensional Dispersive Long Water-Wave System

Hai-Ping Zhu[a], Chun-Long Zheng[a,b], and Jian-Ping Fang[a]

[a] Department of Physics, Zhejiang Lishui University, Lishui Zhejiang 323000, China
[b] Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, Shanghai 200072, China

Reprint requests to C.-L. Z.; Fax: +86-578-2271250; E-mail: zjclzheng@yahoo.com.cn

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Starting from an extended mapping approach, a new type of variable separation solution with arbitrary functions of the (2+1)-dimensional dispersive long water-wave (DLW) system is derived. Then based on the derived solution, we reveal some new types of loop solitons such as bell-like loop solitons and peak-like loop solitons in the (2+1)-dimensional DLW system. – PACS numbers: 05.45.Yv, 03.65.Ge

Key words: Extended Mapping Approach; Dispersive Long Water-Wave System (DLW); Bell-Like Loop Solitons; Peak-Like Loop Solitons.

1. Introduction

In recent studies of higher-dimensional physical models, much effort has been focused on single-valued localized excitations [1 – 4]. However, in various real cases, it is impossible to describe the natural world by single-valued functions only. For instance, there are many complicated folded phenomena such as the folded protein, the folded brain, the folded skin surfaces, and many other kinds of folded biologic systems in the real world. To study these complicated folded natural phenomena is very difficult. Actually, at the present stage, it is impossible to give a complete view on these complicated folded natural phenomena. Fortunately, some research progresses in this aspect have been made and there have been some interesting reports on stable multiple valued solitary waves (folded in all directions) via multiple valued functions [5, 6]. Nevertheless, the real world shows rich and colorful structures and may exhibit intricate structures like semifolded ones, which can be described by neither single valued functions nor multiple valued functions only. For example, some localized structures such as ocean waves may fold in one direction, say x, and localize in a usually single valued way in another direction, say y. In our recent papers [7], we have successfully derived a new type of semifolded localized coherent structure by means of a Painlevé-Bäcklund transformation. Now an important and interesting problem is: Are there similar and/or new semifolded localized coherent structures in other physical models? In particular, can we derive the semifolded localized excitations via other methods such as reduction theory and/or mapping approach [8, 9]? In the present work, our concern is with the new semifolded localized excitations and their evolution properties in (2+1)-dimensions for the dispersive long water-wave (DLW) system

$$u_{yt} + v_{xx} + u_x u_y + u u_{xy} = 0, \tag{1}$$

$$v_t + (uv)_x + u_{xxy} = 0, \tag{2}$$

via a mapping approach. The DLW system was first derived by Boiti et al. [10] as a “weak” lax pair. In [11], Paquin and Winternitz showed that the symmetry algebra of (1) and (2) is infinite-dimensional and is a Kac-Moody-Virasoro structure. Some special similarity solutions are also given in [11] by using symmetry algebra and classical theoretical analysis. The more general symmetry algebra, W_∞, is given in [12]. In [13], Lou gave nine types of two-dimensional similarity reductions and thirteen types of ordinary differential equation reductions. In [14], Lou has shown that the DLW system has no Painlevé property, though the system is Lax or IST integrable. Abundant propagating localized excitations have also been derived by Lou and Zhang [15, 16] with use of the Painlevé analysis and a multilinear variable separation approach. However, to the best of our knowledge, its semifolded localized structures obtained by a mapping approach were not reported in previous literature.

2. New Exact Solutions to the (2+1)-Dimensional DLW System

In this section, we will give a quite general solution to (1) and (2) via an extended mapping approach [17]. For simplicity, we first introduce a variable transformation derived from a standard truncated Painlevé expansion [2]. Substituting $v = u_y$ into (1) and (2) yields the equivalent equation

$$u_{ty} + u_{xxy} + u_x u_y + u u_{xy} = 0. \quad (3)$$

Then taking a balancing procedure similar to [17], we suppose an ansatz for (3):

$$u = f + g\phi(w) + k\phi^{-1}(w), \quad (4)$$

where f, g, k and w are functions of $\{x,y,t\}$ to be determined, ϕ is a function of w solving the Riccati equation: $\frac{\mathrm{d}\phi}{\mathrm{d}w} = \sigma + \phi^2(w)$, where σ is an arbitrary constant. Substituting (4) together with the Riccati equation into (3) and collecting the coefficients of polynomials of ϕ, then setting each coefficient to zero, yields a set of partial differential equations of f, g, k, and w. After calculations with Maple, we obtain exact solutions as follows:

$$\begin{aligned} &f = -\frac{w_{xx} + w_t}{w_x}, \quad g = -2w_x, \quad k = 2\sigma w_x, \\ &w = \chi(x,t) + \varphi(y), \end{aligned} \quad (5)$$

where $\chi \equiv \chi(x,t)$, $\varphi \equiv \varphi(y)$ are two arbitrary variable separated functions of $\{x,t\}$ and y, respectively. Substituting the derived result (5) and the known solutions of the Riccati equation [17] into the above ansatz (4) and the variable transformation relation $v = u_y$, yields a new family of exact solution for the (2+1)-dimensional DLW system.

Case 1. For $\sigma < 0$, we can derive the following solitary wave solutions of the DLW system:

$$\begin{aligned} u_1 = -\Big[&\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}(\chi+\varphi)\right)\chi_t \\ &+ \sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}(\chi+\varphi)\right)\chi_{xx} \\ &+ 2\chi_x^2\sigma\tanh\left(\sqrt{-\sigma}(\chi+\varphi)\right)^2 + 2\sigma\chi_x^2\Big] \\ &\cdot\left[\chi_x\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}(\chi+\varphi)\right)\right]^{-1}, \end{aligned} \quad (6)$$

$$v_1 = 2\frac{\left(\tanh\left(\sqrt{-\sigma}(\chi+\varphi)\right)^2 - 1\right)^2\varphi_y\chi_x\sigma}{\tanh(\sqrt{-\sigma}(\chi+\varphi))^2}, \quad (7)$$

$$\begin{aligned} u_2 = -\Big[&\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}(\chi+\varphi)\right)\chi_t \\ &+ \sqrt{-\sigma}\coth\left(\sqrt{-\sigma}(\chi+\varphi)\right)\chi_{xx} \\ &+ 2\chi_x^2\sigma\coth\left(\sqrt{-\sigma}(\chi+\varphi)\right)^2 + 2\sigma\chi_x^2\Big] \\ &\cdot\left[\chi_x\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}(\chi+\varphi)\right)\right]^{-1}, \end{aligned} \quad (8)$$

$$v_2 = 2\frac{\left(\coth\left(\sqrt{-\sigma}(\chi+\varphi)\right)^2 - 1\right)^2\varphi_y\chi_x\sigma}{\coth\left(\sqrt{-\sigma}(\chi+\varphi)\right)^2}, \quad (9)$$

where $\chi \equiv \chi(x,t)$, $\varphi \equiv \varphi(y)$ are two arbitrary functions of the indicated variables.

Case 2. For $\sigma > 0$, we can obtain the following periodic wave solutions of the DLW system:

$$\begin{aligned} u_3 = \Big[&\tan\left(\sqrt{\sigma}(\chi+\varphi)\right)\chi_t + \tan\left(\sqrt{\sigma}(\chi+\varphi)\right)\chi_{xx} \\ &+ 2\chi_x^2\sqrt{\sigma}\tan\left(\sqrt{\sigma}(\chi+\varphi)\right)^2 - 2\chi_x^2\sqrt{\sigma}\Big] \\ &\cdot\left[\chi_x\tan\left(\sqrt{\sigma}(\chi+\varphi)\right)\right]^{-1}, \end{aligned} \quad (10)$$

$$v_3 = -2\frac{\left(1 + \tan\left(\sqrt{\sigma}(\chi+\varphi)\right)^2\right)^2\varphi_y\chi_x\sigma}{\tan\left(\sqrt{\sigma}(\chi+\varphi)\right)^2}, \quad (11)$$

$$\begin{aligned} u_4 = \Big[&-\cot\left(\sqrt{\sigma}(\chi+\varphi)\right)\chi_t - \cot\left(\sqrt{\sigma}(\chi+\varphi)\right)\chi_{xx} \\ &+ 2\chi_x^2\sqrt{\sigma}\cot\left(\sqrt{\sigma}(\chi+\varphi)\right)^2 - 2\chi_x^2\sqrt{\sigma}\Big] \\ &\cdot\left[\chi_x\cot\left(\sqrt{\sigma}(\chi+\varphi)\right)\right]^{-1}, \end{aligned} \quad (12)$$

$$v_4 = -2\frac{\left(1 + \cot\left(\sqrt{\sigma}(\chi+\varphi)\right)^2\right)^2\varphi_y\chi_x\sigma}{\cot\left(\sqrt{\sigma}(\chi+\varphi)\right)^2}, \quad (13)$$

with two arbitrary functions $\chi(x,t)$ and $\varphi(y)$.

Case 3. For $\sigma = 0$, we can derive the following variable separation solution of the DLW system:

$$u_5 = -\frac{\chi_t\chi + \chi_t\varphi + \chi_{xx}\chi + \chi_{xx}\varphi - 2\chi_x^2}{\chi_x(\chi+\varphi)}, \quad (14)$$

$$v_5 = -2\frac{\varphi_y\chi_x}{(\chi+\varphi)^2}, \quad (15)$$

with two arbitrary functions $\chi(x,t)$ and $\varphi(y)$.

3. Bell-Like and Peak-Like Loop Solitons in the (2+1)-Dimensional DLW System

In this section, we discuss some interesting localized coherent structures for the DLW system. For brevity in the following discussion, we mainly discuss the variable separation solution expressed by v_5 in Case 3, namely

$$v \equiv v_5 = -2\frac{\chi_x \varphi_y}{(\chi+\varphi)^2}. \tag{16}$$

Certainly, one can also discuss the solitary wave solution expressed by v_1 similar to [17], i. e., $v = 2\frac{(\tanh(\sqrt{-\sigma}(\chi+\varphi))^2-1)^2\varphi_y\chi_x\sigma}{\tanh(\sqrt{-\sigma}(\chi+\varphi))^2}$. Because of the arbitrariness of the functions $\chi(x,t)$ and $\varphi(y)$ included in the above solution, the physical field v may possess rich localized structures of coherent solitons [2, 3, 17]. For example, when $\chi(x,t) = kx+ct$ and $\varphi(y) = ly$, where k, c, l are arbitrary constants, we can obtain one of the simplest travelling solitary wave excitations. Meanwhile we may also derive a rich set of non-travelling wave excitations. For instance, when the arbitrary functions are selected to be $\chi(x,t) = \alpha(x)+\tau(t)$ and $\varphi(y) = \gamma(y)$, where α, τ, γ are also arbitrary functions of the indicated arguments, then we can obtain many kinds of non-propagating solitary wave solutions such as dromions, peakons, rings, compactons. Furthermore if $\tau(t)$ is considered to be a periodic function or a solution of a chaotic dynamics as e.g. the Lorenz chaos system, then the non-propagating solitons may possess periodic or chaotic behaviors, which are neglected in the present paper because similar results have been reported already in [17].

3.1. Bell-Like Loop Solitons in the DLW System

Let us pay now our attention to the solution v_5 expressed by (16) and discuss its semifolded localized structures which may exist in certain situations [18], when the arbitrary function φ is chosen to be a suitable single-valued function while the function χ is selected via the relations

$$\begin{aligned}\chi_x &= \sum_{i=1}^{M}\kappa_j(\zeta+d_jt),\\ x &= \zeta+\sum_{i=1}^{M}\varepsilon_j(\zeta+d_jt),\\ \chi &= \int^{\zeta}\chi_x x_\zeta \mathrm{d}\zeta,\end{aligned} \tag{17}$$

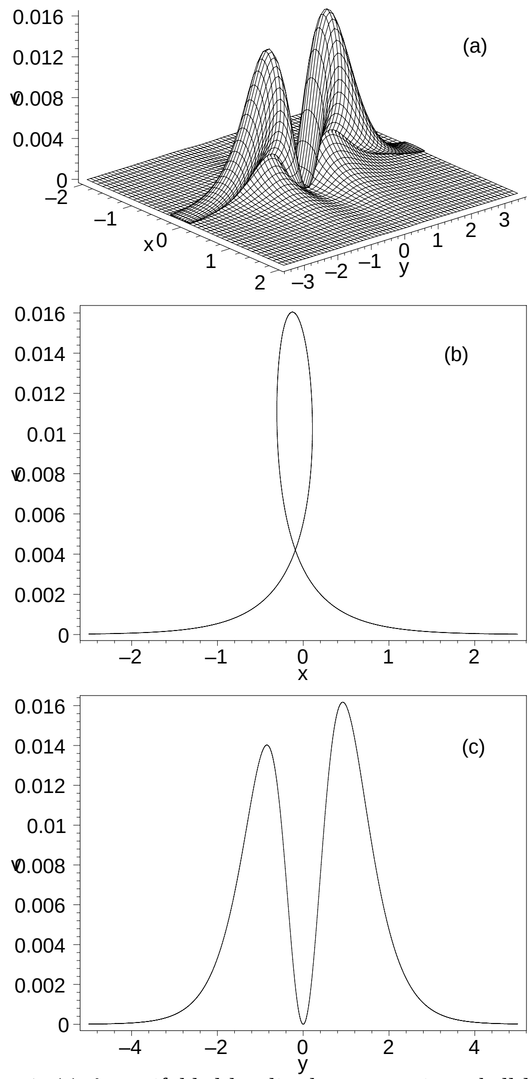

Fig. 1. (a) A semifolded localized structure, i.e, a bell-like loop solition of the physical field v expressed by (16) with condition (18) at time $t = 0$. (b) A sectional view related to (a) at $y = 1$. (c) A sectional view related to (a) at $x = 0$.

where d_j $(j = 1,2,\ldots,M)$ are arbitrary constants and $\{\kappa_j,\varepsilon_j\}$ are localized excitations with properties $\kappa_j(\pm\infty) = 0$, $\varepsilon_j(\pm\infty) = \text{consts}$. From (17), we know that ζ may be a multi-valued function in some suitable regions of x by selecting the functions ε_j appropriately. For instance, when

$$\begin{aligned}\chi_x &= \text{sech}^2(-\zeta+t),\\ x &= -\zeta-1.5\tanh(-\zeta+t),\\ \varphi &= 10-\tanh^3(y),\end{aligned} \tag{18}$$

we can obtain a semifolded localized excitation look-

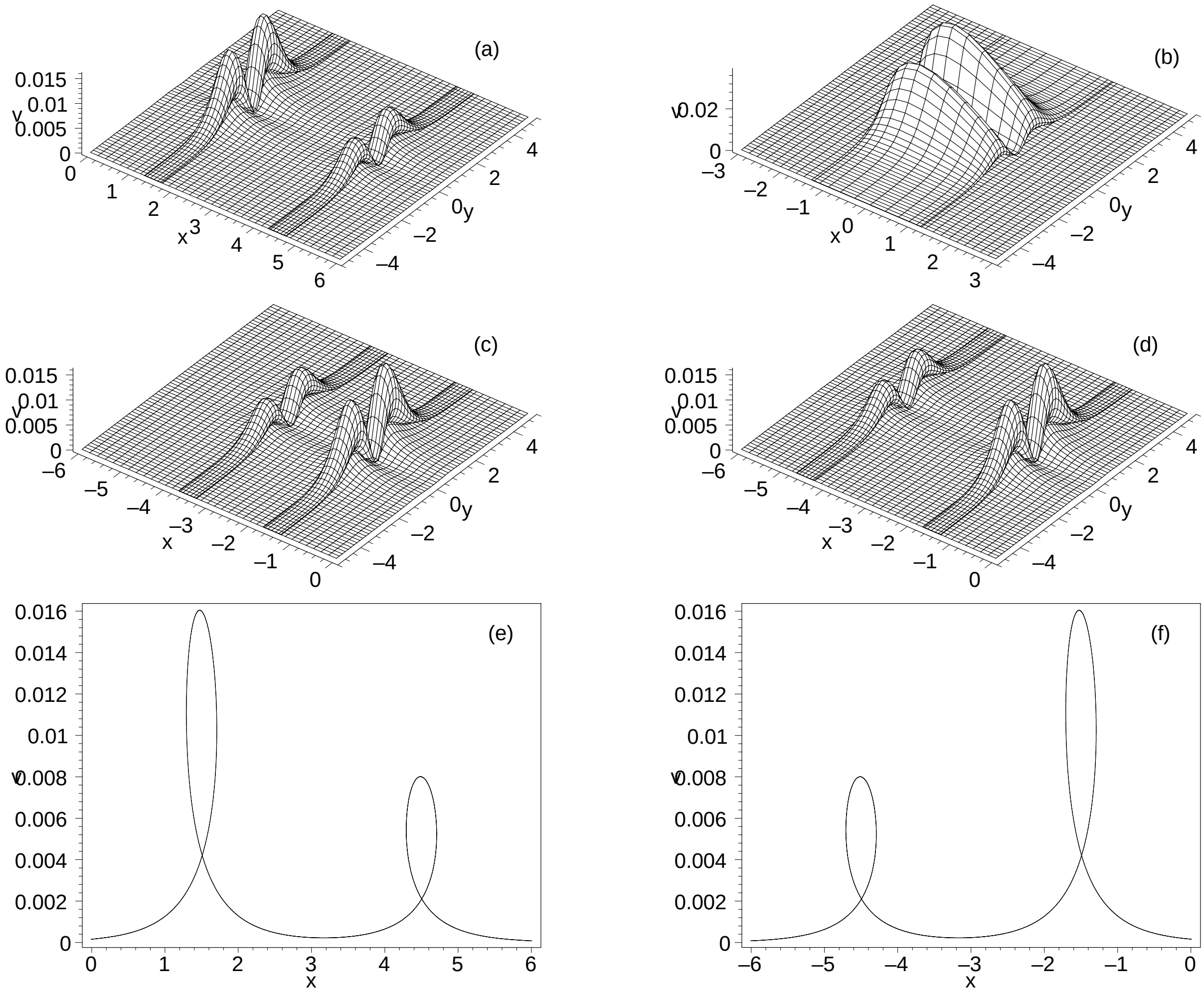

Fig. 2. Evolution of interaction between two bell-like loop solitions of the physical field v expressed by (16) with condition (19) at times: (a) $t=-6$; (b) $t=0$; (c) $t=5$; (d) $t=6$. (e) A sectional view related to (a) at $y=1$. (f) A sectional view related to (d) at $y=1$.

ing like a bell-like loop solition presented in Figure 1. From Fig. 1, one can find that the semifolded localized structure possesses some novel properties, which fold in the x direction and localize in a single-valued way in the y direction like a bell, so we call them bell-like loop solitons.

Let us discuss some interesting evolutional behavior of the novel semifolded solitary waves. For simplicity, we consider a simple case: interaction between two moving loop solitons. For instance, when choosing

$$\chi_x = \mathrm{sech}^2(\zeta) + \mathrm{sech}^2(\zeta - t),$$
$$x = -\zeta + 1.5\tanh(\zeta) + 1.5\tanh(\zeta - t), \qquad (19)$$
$$\varphi = 10 - \tanh^3(y),$$

we can obtain a two-semifolded localized excitation with position shift depicted in Figure 2. In terms of Fig. 2 and by theoretical analysis, we can find that the interaction between the two loop solitons is completely elastic since their amplitudes, velocities and wave shapes do not undergo any change after their collision.

3.2. Peak-Like Loop Solitons in the DLW System

Along the above line, one can obtain some peak-like loop solitons [18] for the field v (16) when χ is chosen an appropriate piecewise smooth function

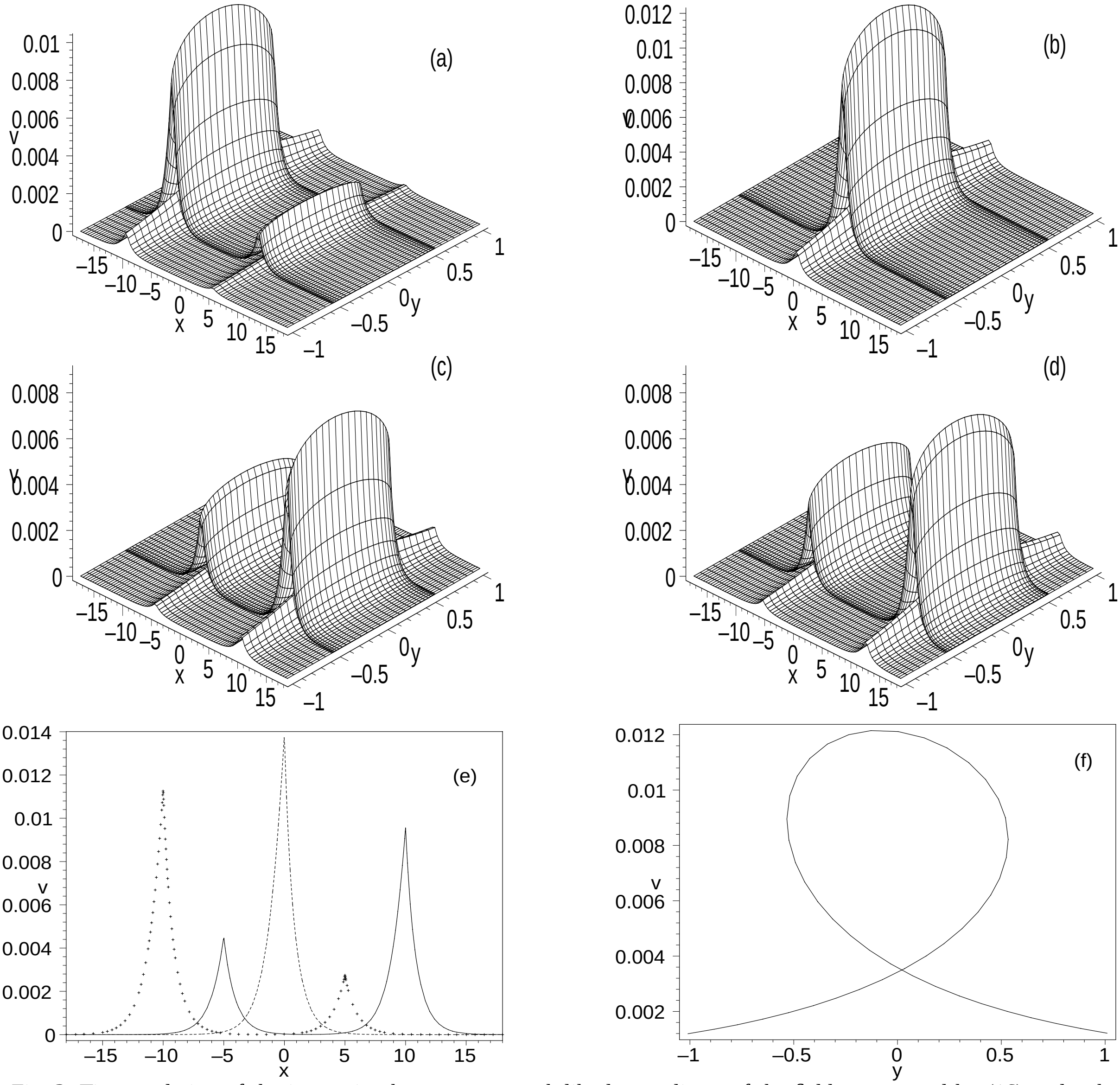

Fig. 3. Time evolution of the interaction between two peak-like loop solitons of the field v expressed by (16) under the conditions (22) at different times: (a) $t = -5$; (b) $t = 0$; (c) $t = 5$; (d) $t = 6$. (e) A sectional view related to (a), (b) and (c) at $y = 0$: dotted line, (a); dashed line, (b); and solid line, (c). (f) A sectional view related to (b) at $x = 0.1$.

of (x,t)

$$\chi = A + \sum_{i=1}^{M} \begin{cases} h_i(x+d_i t), & x+d_i t \le 0, \\ -h_i[-(x+d_i t)] + 2h_i(0), & \\ \quad x+d_i t > 0. & \end{cases} \tag{20}$$

A is an arbitrary constant, while φ is chosen as an appropriate multi-valued function of y:

$$\varphi_y = \sum_{j=1}^{N} g_j(\eta), \quad y = \eta + \sum_{j=1}^{N} Y_j(\eta), \qquad \varphi = \int^{\eta} \varphi_y y_\eta \, \mathrm{d}\eta. \tag{21}$$

Here g_j, Y_j are localized functions of η. From (21), one knows that η may be a multi-valued function in some suitable regions of y by selecting the functions Y_j appropriately. Correspondingly, the function φ_y may be a multi-valued function of y in these ranges. For instance, if taking the arbitrary functions $\chi(x,t)$ and $\varphi(y)$

$$\begin{aligned} &\chi = -20 - \int^x \mathrm{e}^{-|(x+t)|}\mathrm{d}x - 3\int^x \mathrm{e}^{-|(x-2t)|}\mathrm{d}x, \\ &\varphi = \int^\eta \varphi_y y_\eta \mathrm{d}\eta, \\ &\varphi_y = \mathrm{sech}(\theta), \quad y = \theta - 2\tanh(\theta), \end{aligned} \tag{22}$$

for the solution v expressed by (16), then we can find another new type of solitary excitation, a peak-like loop excitation shown in Figure 3. From Fig. 3, one can find that the two new loop solitons possess some interesting properties, which fold in the y direction and localize in a single-valued way in the x direction looking like a peak, so we call them peak-like loop solitons. Moreover, by an analysis similar to Fig. 2, we find that the interaction between the two peak-like loop solitons is not completely elastic since their amplitudes and wave shapes are changed after their collision even though their velocities are completely preserved.

4. Summary and Discussion

In summary, by means of an extended mapping approach, the (2+1)-dimensional DLW system is successfully solved. Based on the derived new type of variable separation solution technique with arbitrary functions, we reveal two new types of localized coherent excitations, bell-like loop soliton and peak-like loop soliton for the DLW system. Some interesting evolutional properties for the bell-like loop solitons and peak-like loop solitons are briefly disscussed. The results show that the interactions between bell-like loop solitons are completely elastic because their amplitudes, velocities and wave shapes are not undergoing any change after collision, while the interactions between peak-like loop solitons are not completely elastic. We expect that the results obtained here may be useful in future studies on the intricate natural world, as well as hope that the approach applied in the present work may be extended in the future to other nonlinear physical systems.

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